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THUE EQUATIONS RELATED TO THE SAME UNIT EQUATION

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Abstract

We answer a question of B. Richter on two quintic Thue equations. This is done by solving the unit equation $\epsilon_1 + \epsilon_2 = 1$ in the maximal real subfield of the 11th cyclotomic field. We further investigate which other Thue equations lead to the same unit equation, and thus have been solved as well.

1 Introduction

In [R, Sections IX, X], B. Richter asks for determining the complete sets of solutions of the quintic Thue equations

$$U^5 - 55U^3V^2 + 253U^2V^3 - 429UV^4 + 253V^5 = 1, \tag{1}$$

which has the solutions $(U, V) = (-8,-3), (-2,-1), (1,0), (3,1), (5,2)$, and no others with $|V| \leq 1000$, and

$$14641U^5 - 6655U^3V^2 + 2783U^2V^3 - 429UV^4 + 23V^5 = 1, \tag{2}$$

which has the solution $(U, V) = (1, 4)$, and no others with $|V| \leq 1000$. In this note we answer this question, in proving that there are no other solutions.

The method we use is that of [TdW1]. We reduce the Thue equations to a unit equation, and then bound the variables in this unit equation by transcendence theory, thus making the problem finite. However, the upper bound found in this way is extremely large, so we have to use computational diophantine approximation (based on the LLL-algorithm) to find a much better upper bound, which is small enough to admit enumeration of the remaining cases.

A surprising fact is that both Thue equations (1) and (2) give rise to the same unit equation. This unit equation is

$$\epsilon_1 + \epsilon_2 = 1, \tag{3}$$

where the unknowns ϵ_1, ϵ_2 are units in the quintic field $\mathbb{K} = \mathbb{Q}(e^{2\pi i/11} + e^{-2\pi i/11})$, the maximal real subfield of the 11th cyclotomic field, which happens to be the totally real quintic field with minimal discriminant. Below we will determine the complete set of solutions of equation (3).

The question now arises which other Thue equations $F(X,Y) = M$ (where $F(X,Y) \in \mathbb{Z}[X,Y]$ of degree 5, and $M \in \mathbb{Z}$) give rise to the same unit equation (3). This question is of some interest, because we get the complete sets of solutions for these Thue equations for free, since we have solved (3) anyway. We answer this question for the case that M is not divisible by a splitting prime.

The field $\mathbb{K}$ is generated over $\mathbb{Q}$ by $\theta = -2\cos(\frac{2}{11}\pi)$, which is a root of $x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$. The field discriminant is 11^4, the field is Galois, has trivial class group, and a set of fundamental units is $\{\theta, \sigma(\theta), \sigma^2(\theta), \sigma^3(\theta)\}$, where σ is a generator of Gal($\mathbb{K}$). We take σ to be the automorphism with $\sigma(\theta) = 2 - \theta^2$.

The set of automorphisms acting on the set of solutions (ϵ_1, ϵ_2) of (3) is generated by $(\epsilon_1, \epsilon_2) \to (\sigma(\epsilon_1), \sigma(\epsilon_2))$, $(\epsilon_1, \epsilon_2) \to (\epsilon_2, \epsilon_1)$ and $(\epsilon_1, \epsilon_2) \to (1/\epsilon_2, -\epsilon_1/\epsilon_2)$. Thus the solution set can be divided into equivalence classes of 30 solutions. We write solutions as

$$(\epsilon_1, \epsilon_2) = \left(\pm\theta^a(\sigma(\theta))^b(\sigma^2(\theta))^c(\sigma^3(\theta))^d, \pm\theta^e(\sigma(\theta))^f(\sigma^2(\theta))^g(\sigma^3(\theta))^h\right).$$

Our main result is the following. Section 2 is devoted to a proof of this result.

Theorem 1 *There are exactly* 570 *solutions of equation* (3)*, divided into* 19 *equivalence classes of* 30 *solutions each. Table* 1 *below contains one arbitrary representative of each equivalence class.*

ϵ_1					ϵ_2				
$\pm$	a	b	c	d	$\pm$	e	f	g	h
+	2	4	−1	−3	+	4	2	3	−1
+	3	2	4	−2	+	3	0	−2	−3
+	2	−3	−3	−4	+	1	0	1	−4
+	1	0	−2	2	−	0	2	−3	−1
−	1	0	2	−3	+	1	1	−2	−4
+	2	−2	−2	−2	−	−1	−2	−1	−3
+	2	−2	0	1	−	0	−1	2	2
−	2	2	2	−1	+	3	0	−1	−1
−	1	2	−1	−2	+	2	0	0	−1
−	1	0	2	−1	−	2	−1	−1	−1
+	0	1	−1	1	−	1	−1	−2	0
+	1	2	0	0	−	2	0	−1	0
−	1	1	−1	−2	+	2	1	2	−1
+	1	1	−1	0	−	0	1	0	2
+	1	−1	−1	0	+	−1	−1	−1	0
+	1	1	1	−1	+	1	−1	−1	−1
−	1	0	−1	1	+	0	1	−1	0
−	0	1	0	0	+	−1	0	0	1
+	1	0	0	0	−	0	0	−1	−1

Table 1: Representatives of the 19 equivalence classes of solutions of the unit equation (3).

The following two theorems treat Thue equations $F(X,Y) = 11^s$ with $s \leq 4$. Notice that the Prime Ideal Removing Lemma of [TdW2] guarantees that there are no solutions of these equations with $s \geq 5$ and $\gcd(X,Y) = 1$. That's why we call these equations Thue equations rather than Thue-Mahler equations.

Theorem 2 *The Thue equation*

$$X^5 - 3X^4Y - 14X^3Y^2 + 15X^2Y^3 + XY^4 - Y^5 = 11^s \tag{4}$$

in $(X,Y) \in \mathbb{Z}^2$, *with* $s \in \{0,1,2,3,4\}$, *has only the solutions* $(X,Y,s) = (1,0,0)$, $(0,-1,0)$, $(-1,-1,0)$, $(-1,2,2)$, $(-5,-1,2)$.

Theorem 3 *The Thue equation*

$$X^5 - X^4Y - 4X^3Y^2 + 3X^2Y^3 + 3XY^4 - Y^5 = 11^s \tag{5}$$

in $(X,Y) \in \mathbb{Z}^2$, *with* $s \in \{0,1,2,3,4\}$, *has only the solutions* $(X,Y,s) = (1,0,0)$, $(0,-1,0)$, $(1,1,0)$, $(-1,1,0)$, $(2,1,0)$, $(2,-1,1)$.

In Section 3 we will see that Theorems 2 and 3 are consequences of Theorem 1.

Notice that Richter's equation (1) leads to (5) with $s = 0$ by the transformation $(U,V) = (3X-5Y, X-2Y)$, and that Richter's equation (2) leads to (5) with $s = 1$ by the transformation $(U,V) = ((3X-5Y)/11, X-2Y)$. Thus Richter's equations have both been completely solved with Theorem 3.

Further notice that our Theorem 3 with $s = 0$ is not new: it was proved recently by Paul Voutier [V], who used essentially the same method as we'll do. In fact, he came very close to proving our Theorem 1.

We also show that all the Thue equations $F(U,V) = M$, where M is not divisible by a splitting prime, and that one can solve by our method using Theorem 1, can be reduced by a bilinear transformation of the variables to one of the two Thue equations (4) and (5). Indeed, we have the following result, which also itself is a consequence of Theorem 1. A bilinear transformation of variables $(U,V) = (pX+qY, rX+sY)$ is called *unimodular* if $ps - qr = \pm 1$, and 11-*modular* if $ps - qr = \pm 11^t$ for some nonzero integer t.

Theorem 4 *Let* $F(X,Y) \in \mathbb{Z}[X,Y]$ *be irreducible such that* $F(t,1) = 0$ *has a nonrational root* $\psi \in \mathbb{K}$. *Let* $M \in \mathbb{Z}$ *be nonzero. If for some* (k,ℓ) *with* $1 \le k < \ell \le 4$ *both* $\frac{\psi - \sigma^\ell(\psi)}{\psi - \sigma^k(\psi)}$ *and* $\frac{\sigma^k(\psi) - \sigma^\ell(\psi)}{\psi - \sigma^k(\psi)}$ *are units, and if* M *is not divisible by a prime that splits in* $\mathbb{K}$, *then the Thue equation* $F(U,V) = M$ *is equivalent to* (4) *or* (5) *by a unimodular or* 11-*modular transformation of the variables.*

We have not been able to deal completely with the case where M is divisible by a splitting prime. We cannot exclude the possibility of finding a $\mu \in \mathbb{K}$ with $M = \mathrm{Norm}_{\mathbb{K}/\mathbb{Q}}(\mu)$ such that μ and $\sigma(\mu)$ are not associated, yet with both $\frac{\psi - \sigma^\ell(\psi)}{\psi - \sigma^k(\psi)} \frac{\sigma^k(\mu)}{\sigma^\ell(\mu)}$ and $\frac{\sigma^k(\psi) - \sigma^\ell(\psi)}{\psi - \sigma^k(\psi)} \frac{\mu}{\sigma^\ell(\mu)}$ being units for some (k,ℓ) with $1 \le k < \ell \le 4$. On the other hand, we also did not find such examples. We feel not certain enough to make any conjecture here.

We finish the paper with a short section on the concept of Thue graphs, which seems to be new.

2 Solving the unit equation

We start with a lemma on general units in $\mathbb{K}$. Let $\epsilon \in \mathbb{K}$ be any unit, and put

$$\epsilon = \pm\theta^a(\sigma(\theta))^b(\sigma^2(\theta))^c(\sigma^3(\theta))^d$$

for $a, b, c, d \in \mathbb{Z}$, and

$$A = \max\{|a|, |b|, |c|, |d|\}.$$

For a matrix $M = (m_{i,j})$ (where i is the row counter and j the column counter) we introduce the *row norm*

$$\mathbb{N}[M] = \max_i \sum_j |m_{i,j}|.$$

For $k = 0, 1, 2, 3, 4$ put

$$M_k = \left(\log\left|\sigma^{i+j \pmod 5}(\theta)\right|\right)_{\substack{i\,=\,0,\ldots,4,\, i\neq k \\ j\,=\,0,\ldots,3}},$$

and let $C_1 = \min_{k=0,\ldots,4} \mathbb{N}[M_k^{-1}]$.

Lemma 1 *There is an $i_0 \in \{0, 1, 2, 3, 4\}$ such that*

$$\left|\sigma^{i_0}(\epsilon)\right| \leq \exp\left(-\frac{1}{4C_1}A\right).$$

Proof of Lemma 1 Note that for $k \in \{0, 1, 2, 3, 4\}$ we have

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = M_k^{-1} \begin{pmatrix} \vdots \\ \log|\sigma^i(\epsilon)| \\ \vdots \end{pmatrix}_{i=0,\ldots,4,i\neq k}.$$

Hence, choosing k such that $\mathbb{N}[M_k^{-1}]$ is minimal, we obtain

$$A \leq C_1 \max_i \left|\log\left|\sigma^i(\epsilon)\right|\right|.$$

Now let i_0, i_1 be such that

$$\left|\sigma^{i_0}(\epsilon)\right| = \min_i \left|\sigma^i(\epsilon)\right|, \quad \left|\sigma^{i_1}(\epsilon)\right| = \max_i \left|\sigma^i(\epsilon)\right|.$$

Because $\prod_i |\sigma^i(\epsilon)| = 1$ we have $|\sigma^{i_0}(\epsilon)| \leq 1 \leq |\sigma^{i_1}(\epsilon)|$, and it follows that

$$\text{either} \quad \max_i \left|\log\left|\sigma^i(\epsilon)\right|\right| = \left|\log\left|\sigma^{i_1}(\epsilon)\right|\right| \quad \text{or} \quad \max_i \left|\log\left|\sigma^i(\epsilon)\right|\right| = -\left|\log\left|\sigma^{i_0}(\epsilon)\right|\right|.$$

In the first case we use again $\prod_i |\sigma^i(\epsilon)| = 1$, and obtain

$$\left|\sigma^{i_0}(\epsilon)\right| \leq \prod_{i\neq i_1} \left|\sigma^i(\epsilon)\right|^{1/4} = \left|\sigma^{i_1}(\epsilon)\right|^{-1/4} \leq \exp\left(-\frac{1}{4C_1}A\right),$$

and in the second case we find at once

$$\left|\sigma^{i_0}(\epsilon)\right| \leq \exp\left(-\frac{1}{C_1}A\right) \leq \exp\left(-\frac{1}{4C_1}A\right).$$

□

From now on, let (ϵ_1, ϵ_2) be a solution of the unit equation (3), and put

$$\epsilon_1 = \pm\theta^a(\sigma(\theta))^b(\sigma^2(\theta))^c(\sigma^3(\theta))^d, \quad \epsilon_2 = \pm\theta^e(\sigma(\theta))^f(\sigma^2(\theta))^g(\sigma^3(\theta))^h,$$

$$A = \max\{|a|, |b|, |c|, |d|, |e|, |f|, |g|, |h|\}.$$

Then $A > 0$. Further we put for $k = 0, 1, 2, 3, 4$

$$\Lambda_k = a\log\left|\sigma^k(\theta)\right| + b\log\left|\sigma^{k+1}(\theta)\right| + c\log\left|\sigma^{k+2}(\theta)\right| + d\log\left|\sigma^{k+3}(\theta)\right|,$$

then $\Lambda_k \neq 0$ for all k.

Lemma 2 *If $A \geq 10$ then there is a $k \in \{0, 1, 2, 3, 4\}$ such that*

$$|\Lambda_k| < 1.18892e^{-0.121703A}. \tag{6}$$

Proof of Lemma 2 We may assume without loss of generality that $A = \max\{|e|, |f|, |g|, |h|\}$. Then according to Lemma 1 there is a $k \in \{0, 1, 2, 3, 4\}$ such that

$$\left|\sigma^k(\epsilon_1) - 1\right| = \left|\sigma^k(\epsilon_2)\right| \leq \exp\left(-\frac{1}{4C_1}A\right),$$

and one easily computes $C_1 < 2.05418$. By $\sigma^k(\epsilon_1) = e^{\Lambda_k}$ we thus have $\left|e^{\Lambda_k} - 1\right| < e^{-0.121703A}$. If $A \geq 10$ then $e^{-0.121703A} < 0.3$, hence $|\Lambda_k| < 1.18892\left|e^{\Lambda_k} - 1\right|$, and the Lemma follows. □

Lemma 3 $A < 3.68384 \times 10^{20}$.

Proof of Lemma 3 Since $\Lambda_k \neq 0$, the main result of Baker and Wüstholz [BW], with $n = 4, D = 5, h'(\sigma^k(\theta)) = \frac{1}{5}\log|\theta\sigma^2(\theta)\sigma^4(\theta)| < 0.288380$ for $k = 0, 1, 2, 3, 4$, and $h'(L) = \log A$, yields

$$|\Lambda_k| > \exp(-9.46737 \times 10^{17}\log A).$$

Combining this with Lemma 2, the Lemma follows at once. □

For a large constant C we consider the lattice $\Gamma = \{\mathcal{A}\boldsymbol{x} | \boldsymbol{x} \in \mathbb{Z}^4\}$, where

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ [C\log|\theta|] & [C\log|\sigma(\theta)|] & [C\log|\sigma^2(\theta)|] & [C\log|\sigma^3(\theta)|] \end{pmatrix}$$

(here $[\cdot]$ stands for rounding to an integer). Let

$$d(\Gamma) = \min_{\boldsymbol{x}\in\mathbb{Z}^4, \boldsymbol{x}\neq \boldsymbol{0}} |\mathcal{A}\boldsymbol{x}|,$$

which is the minimal distance between two lattice points. The LLL-algorithm can be used to compute a lower bound for it. This lower bound can be expected to be of the size of $C^{1/4}$.

The following lemma is a variant of [dW, Lemma 3.7].

Lemma 4 *If $d(\Gamma) > \sqrt{61}\,A_0$ then there are no solutions of (6) satisfying*

$$\left\lfloor \frac{0.173046 + \log C - \log\left(\sqrt{d(\Gamma)^2 - 12A_0^2} - 7A_0\right)}{0.121703} \right\rfloor < A \leq A_0.$$

Proof of Lemma 4 Assume we have a solution of (6) satisfying $A \leq A_0$. Then there are $a_1, b_1, c_1, d_1 \in \mathbb{Z}$ with

$$\begin{aligned} \Lambda_k &= a\log\left|\sigma^k(\theta)\right| + b\log\left|\sigma^{k+1}(\theta)\right| + c\log\left|\sigma^{k+2}(\theta)\right| + d\log\left|\sigma^{k+3}(\theta)\right| \\ &= a_1\log|\theta| + b_1\log|\sigma(\theta)| + c_1\log\left|\sigma^2(\theta)\right| + d_1\log\left|\sigma^3(\theta)\right|, \end{aligned}$$

and it's easy to see that

$$\max\{|a_1|, |b_1|, |c_1|, |d_1|\} \leq 2A, \quad |a_1| + |b_1| + |c_1| + |d_1| \leq 7A,$$

whatever k is. We now consider the lattice point $\mathcal{A}(a_1, b_1, c_1, d_1)^\top = (a_1, b_1, c_1, \lambda)^\top$, with

$$\lambda = a_1[C\log|\theta|] + b_1[C\log|\sigma(\theta)|] + c_1[C\log\left|\sigma^2(\theta)\right|] + d_1[C\log\left|\sigma^3(\theta)\right|].$$

This lattice point is nonzero since $\Lambda_k \neq 0$. We have on the one hand

$$d(\Gamma)^2 \leq a_1^2 + b_1^2 + c_1^2 + \lambda^2 \leq 12A^2 + \lambda^2,$$

and on the other hand

$$|\lambda - C\Lambda_k| \leq |a_1| + |b_1| + |c_1| + |d_1| \leq 7A.$$

It follows by $A \leq A_0$ that

$$|\Lambda_k| \geq \frac{1}{C}\left(\sqrt{d(\Gamma)^2 - 12A_0^2} - 7A_0\right).$$

The right hand side is positive since $d(\Gamma) > \sqrt{61}\,A_0$, and the result follows by (6). □

Proof of Theorem 1 Initially, with $A_0 = 3.68384 \times 10^{20}$ we take $C = 10^{88}$, which will turn out to be sufficient. Indeed, with $C = 10^{88}$ we found $d(\Gamma) > 3.52058 \times 10^{21} > \sqrt{61}\,A_0$, and hence by Lemmas 3 and 4 we find $A \leq 1271$.

With $A_0 = 1271$ we take $C = 10^{19}$, which leads to $d(\Gamma) > 11468.7 > \sqrt{61}\,A_0$, and hence we reach $A \leq 299$.

With $A_0 = 299$ we take $C = 10^{16}$, which leads to $d(\Gamma) > 3262.54 > \sqrt{61}\, A_0$, and hence we reach $A \leq 247$.

With $A_0 = 247$ we take again $C = 10^{16}$, now leading to $A \leq 244$.

We did a direct search for solutions of (6) with $41 \leq A \leq 244$. We found 3338 solutions, which we checked for coming from a solution of (3). None did.

Finally we did a direct search for solutions of (3) with $A \leq 40$, and found 570 solutions. This proves the theorem. □

Notice that the largest value of A that occurs is 7.

3 Thue equations related to the unit equation

In this Section we prove Theorems 2, 3 and 4, starting with the last one.

Proof of Theorem 4 Let $F(X,Y) \in \mathbb{Z}[X,Y]$ be an irreducible binary form of degree 5, such that a root ψ of $F(t,1) = 0$ is in $\mathbb{K} = \mathbb{Q}(\theta)$. The Thue equation $F(U,V) = M$ is solvable only if there is a $\mu \in \mathbb{K}$ such that $M = \mathrm{Norm}_{\mathbb{K}/\mathbb{Q}}(\mu)$. Then it is equivalent to

$$U - V\psi = \mu\epsilon,$$

where ϵ is a unit of $\mathbb{K}$.

Applying σ^k and σ^ℓ to the above equation for some $1 \leq k < \ell \leq 4$, and eliminating U and V from the three conjugate equations we then have, we find

$$\left(\sigma^k(\psi) - \sigma^\ell(\psi)\right)\mu\epsilon + \left(\sigma^\ell(\psi) - \psi\right)\sigma^k(\mu)\sigma^k(\epsilon) + \left(\psi - \sigma^k(\psi)\right)\sigma^\ell(\mu)\sigma^\ell(\epsilon) = 0.$$

Hence we find the unit equation

$$\frac{\psi - \sigma^\ell(\psi)}{\psi - \sigma^k(\psi)}\,\frac{\sigma^k(\mu)}{\sigma^\ell(\mu)}\,\frac{\sigma^k(\epsilon)}{\sigma^\ell(\epsilon)} - \frac{\sigma^k(\psi) - \sigma^\ell(\psi)}{\sigma^k(\psi) - \psi}\,\frac{\mu}{\sigma^\ell(\mu)}\,\frac{\epsilon}{\sigma^\ell(\epsilon)} = 1\,.$$

Notice that this unit equation is equation (3) if and only if both

$$\eta_1 = \frac{\psi - \sigma^\ell(\psi)}{\psi - \sigma^k(\psi)}\,\frac{\sigma^k(\mu)}{\sigma^\ell(\mu)} \quad \text{and} \quad \eta_2 = -\frac{\sigma^k(\psi) - \sigma^\ell(\psi)}{\sigma^k(\psi) - \psi}\,\frac{\mu}{\sigma^\ell(\mu)} \tag{7}$$

are units, and that they satisfy the unit equation

$$\frac{\sigma^\ell(\mu)}{\sigma^k(\mu)}\,\eta_1 + \frac{\sigma^\ell(\mu)}{\mu}\,\eta_2 = 1\,. \tag{8}$$

If p is a prime dividing M then either it remains prime in $\mathbb{K}$, or it ramifies completely, in which case $(p) = (11) = (\pi_{11})^5$, where $\pi_{11} = 2 - \theta - \theta^2$. In both cases $\frac{\sigma^\ell(\mu)}{\sigma^k(\mu)}$ and $\frac{\sigma^\ell(\mu)}{\mu}$ are units, and the equation (8) is just the unit equation (3). Notice that the Prime Ideal Removing

Lemma of [TdW2] shows that the condition $\gcd(U, V) = 1$ implies that $M = \pm 11^s$ for an $s \in \{0, 1, 2, 3, 4\}$. Here the sign is irrelevant.

Now for each of the 570 solutions (ϵ_1, ϵ_2) to (3), for each $s \in \{0, 1, 2, 3, 4\}$, and for each (k, ℓ) with $1 \leq k < \ell \leq 4$, we determine η_1, η_2 from (8). Then we write out the equations (7) in terms of the integral basis, and try to solve them for ψ. Notice that if we write $\psi = x + y\theta + z\theta^2 + u\theta^3 + v\theta^4$ for unknown $x, y, z, u, v \in \mathbb{Z}$, then x is arbitrary (the transformation $\psi \to \psi + x_0$ for an $x_0 \in \mathbb{Z}$ corresponds to the unimodular transformation of variables $(U, V) = (X + x_0 Y, Y)$), and that solutions are determined up to multiplication by a rational number. Of course we take the smallest multiplication factor such that ψ becomes integral.

We thus have a homogeneous linear system of 5 equations and 4 unknowns, and this has a nontrivial solution if and only if the corresponding coefficient matrix has rank ≤ 3. We checked this in all our cases, and for each pair (k, ℓ) we found 55 cases of rank 3, and no cases of rank < 3. Because of the obvious action of the Galois group, this leads to 11 quintuples of conjugated ψ's. For each of these quintuples we took a representative ψ with x such that the corresponding minimal polynomial has minimal height. We searched for unimodular and 11-modular transformations between these 11 polynomials, and this yielded results as in Table 2 below. Here, the column 'relation' shows how each ψ can be expressed by a unimodular or 11-modular transformation in either ψ_1 or ψ_2.

minimal polynomial	root ψ	relation
$t^5 - 3t^4 - 14t^3 + 15t^2 + t - 1$	$2 - 6\theta - \theta^2 + 2\theta^3$	$\psi = \psi_1$
$t^5 + 76t^4 - 37t^3 - 30t^2 + 14t - 1$	$-16 + 69\theta - 35\theta^2 - 25\theta^3 + 14\theta^4$	$\psi = \frac{2\psi_1+1}{-\psi_1+5}$
$t^5 + 13t^4 - 27t^3 + 9t^2 + 4t - 1$	$-1 + \theta + 5\theta^2 - \theta^3 - 2\theta^4$	$\psi = \frac{\psi_1}{\psi_1-1}$
$t^5 + 9t^4 - 16t^3 - 53t^2 + 37t + 23$	$-2 - 2\theta + 5\theta^2 + 2\theta^3 - 2\theta^4$	$\psi = \frac{3\psi_1-4}{2\psi_1+1}$
$t^5 + t^4 - 15t^3 - 14t^2 + 3t + 1$	$-4 - \theta + 5\theta^2 - \theta^4$	$\psi = -\frac{1}{\psi_1}$
$t^5 - t^4 - 4t^3 + 3t^2 + 3t - 1$	$-3\theta + \theta^3$	$\psi = \psi_2$
$t^5 + 35t^4 + 127t^3 - 200t^2 - 95t + 109$	$1 - 17\theta - 21\theta^2 + 4\theta^3 + 6\theta^4$	$\psi = \frac{4\psi_2-3}{\psi_2+2}$
$t^5 + 10t^4 - 15t^3 + 3t^2 + 3t - 1$	$-2 + 11\theta - 5\theta^2 - 4\theta^3 + 2\theta^4$	$\psi = \frac{\psi_2-1}{\psi_2-2}$
$t^5 + 6t^4 - t^3 - 10t^2 - 6t - 1$	$-6 - 2\theta + 8\theta^2 + \theta^3 - 2\theta^4$	$\psi = -\frac{1}{\psi_2+1}$
$t^5 - 2t^4 - 5t^3 + 2t^2 + 4t + 1$	$-2\theta + \theta^3$	$\psi = \frac{1}{\psi_2-1}$
$t^5 - 3t^4 - 3t^3 + 4t^2 + t - 1$	$-1 - \theta + \theta^2$	$\psi = \frac{1}{\psi_2}$

Table 2: The possibilities for ψ and their interrelations.

Now notice that ψ_1 and ψ_2 correspond to the Thue equations (4) and (5) respectively, so that we have proved Theorem 4. □

Proof of Theorems 2 and 3 For each of the 570 solutions (ϵ_1, ϵ_2) of the unit equation (3), for both $\psi = \psi_1, \psi_2$, for each $s \in \{0, 1, 2, 3, 4\}$, and for each (k, ℓ) with $1 \leq k < \ell \leq 4$, we must check whether there exists a unit ϵ satisfying

$$\epsilon_1 = \frac{\psi - \sigma^\ell(\psi)}{\psi - \sigma^k(\psi)} \left(\frac{\sigma^k(\pi_{11})}{\sigma^\ell(\pi_{11})}\right)^s \frac{\sigma^k(\epsilon)}{\sigma^\ell(\epsilon)}, \quad \epsilon_2 = -\frac{\sigma^k(\psi) - \sigma^\ell(\psi)}{\sigma^k(\psi) - \psi} \left(\frac{\pi_{11}}{\sigma^\ell(\pi_{11})}\right)^s \frac{\epsilon}{\sigma^\ell(\epsilon)}.$$

Written out in terms of the fundamental units this yields a linear system of 8 equations in 4 unknowns. We solved all these systems, and this produced only the mentioned solutions. □

4 Thue graphs

We noticed the following interesting phenomenon. Take a Thue equation $F(X,Y) = M$ with a solution $(X,Y) \in \mathbb{Z}^2$. Let $\mathbb{K} = \mathbb{Q}(\theta)$, with $F(\theta, 1) = 0$. The solution of our Thue equation gives rise to a solution (ϵ_1, ϵ_2) in units of some field $\mathbb{L}$ (which is between $\mathbb{K}$ and its normal closure), of a certain unit equation

$$\nu_1\epsilon_1 + \nu_2\epsilon_2 = 1,$$

where ν_1, ν_2 are parameters in $\mathbb{L}$, not necessarily integral, and with norm 1.

Now we can connect this Thue equation with solution to another Thue equation with solution. Namely, put $\psi = \nu_1\epsilon_1$, and let μ be the denominator of ν_2. Then $(X,Y) = (1,1)$ is a solution of the Thue equation

$$G(X,Y) = \prod_{\sigma\in\mathrm{Gal}(\mathbb{L}/\mathbb{Q})} \sigma(\mu)(X - Y\sigma(\psi)) = \pm\mathrm{Norm}_{\mathbb{L}/\mathbb{Q}}(\mu).$$

If the field $\mathbb{L}$ is chosen properly, $G(X,Y) \in \mathbb{Z}[X,Y]$ will be irreducible.

Now we can iterate this procedure of connecting Thue equations with solutions to Thue equations with solutions, and draw a graph showing these connections. Then we can study what happens in these *Thue graphs*. For example, do cycles occur, and how often? Experiments might shed some light on what will happen. We performed only a very small experiment with the solutions of the unit equation (3) that we have found in Theorem 1, stopping the iteration as soon as we reached a unit equation not equivalent to (3). We found no cycles, only a number of not very deep trees.

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